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The random cluster model and a new integration identity

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Abstract

We evaluate the free energy of the random cluster model at its critical point for $0 < q < 4$ using an exact result due to Baxter, Temperley and Ashley, and obtain an explicit expression in the form of an infinite series. It is found that the resulting series expression assumes a form which depends on whether or not $\pi/2\cos^{-1}(\sqrt{q}/2)$ is a rational number. As a by-product, our consideration leads to a closed-form evaluation of the integral

$$\frac{1}{4\pi^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln[A + B + C - A\cos\theta - B\cos\phi - C\cos(\theta+\phi)]$$
$$= -\ln(2S) + (2/\pi)[\mathrm{Ti}_2(AS) + \mathrm{Ti}_2(BS) + \mathrm{Ti}_2(CS)],$$

which arises in lattice statistics, where $A, B, C \geqslant 0$ and $S = 1/\sqrt{AB + BC + CA}$.

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The q-state Potts model, proposed in 1954 by Potts [1] as a model of generalized order–disorder transitions, has remained to this day as one of the most outstanding unsolved problems in statistical mechanics. While the model was originally proposed for integral q, in 1969 Kasteleyn and Fortuin [2, 3] introduced the notion of a random cluster model which extends its consideration to arbitrary q. The random cluster model has since played an important role in numerous other frontiers including lattice statistics, graph theory and combinatorics [3–5].

The solution of the random cluster model is not known for general q. In a remarkable paper published in 1978, Baxter, Temperley and Ashley [6] used an earlier result due to Kelland [7] on a 20-vertex model to derive a number of exact results on the random cluster model including an explicit expression of its free energy at the critical point. Specifically, they deduced the critical free energy in the form of an infinite series for $q > 4$ and in the form of a definite integral for $q < 4$.

The regime $q \leqslant 4$ is of particular interest as it is the regime in which the random cluster model exhibits a second-order transition [8] mimicking many order–disorder transitions occurring in nature. Hence, it warrants a closer examination and here we explore in this direction.

We carry out the integration in the Baxter–Temperley–Ashley expression and recast the solution in the form of a series. We find the resulting series to assume different forms depending on an analytic property of q, which is perhaps an indicative of a salient nature of the (yet unknown) general solution. As a by-product, our analysis, when compared with the known Ising ($q = 2$) solution, yields a closed-form evaluation of the integral

$$I(A, B, C) = \frac{1}{4\pi^2}\int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \ln[A + B + C - A\cos\theta - B\cos\phi - C\cos(\theta + \phi)], \tag{1}$$

which arises often in lattice-statistical problems. The integral has not previously been evaluated except in special cases when it evaluates spanning tree entropies as given in (30) below.

We first describe the random cluster model [2].

Consider a triangular lattice of N_s sites. The partition function of a random cluster model on the lattice is the graph generating function [2, 3]

$$Z_{N_s}^{\mathrm{RC}}(q; K_1, K_2, K_3) = \sum_S q^{c(S)}(\mathrm{e}^{K_1} - 1)^{\ell_1(S)}(\mathrm{e}^{K_2} - 1)^{\ell_2(S)}(\mathrm{e}^{K_3} - 1)^{\ell_3(S)}, \tag{2}$$

where the summation is over all edge sets S of the triangular lattice, $c(S)$ is the number of connected clusters in S including isolated points, $\ell_\alpha(S)$ is the number of lines in S in the direction $\alpha = 1, 2, 3$ and q is a variable which is an extension of the number of states of the Potts model. For $q =$ integers, the graph generating function (2) coincides with the partition function of a q-state Potts model with (reduced) interactions K_1, K_2, K_3 on the same lattice. One is interested in the evaluation of the per-site 'free energy'

$$f^{\mathrm{RC}}(q) = \lim_{N_s\to\infty} N_s^{-1} \ln Z_{N_s}^{\mathrm{RC}}(q; K_1, K_2, K_3). \tag{3}$$

The random cluster model is known to be critical at [6, 9, 10]

$$\sqrt{q}x_1x_2x_3 + x_1x_2 + x_2x_3 + x_3x_1 = 1, \tag{4}$$

where

$$x_\alpha = (\mathrm{e}^{K_\alpha} - 1)/\sqrt{q} \geqslant 0. \tag{5}$$

For $0 < q < 4$, define $\phi(q)$ and $v_\alpha(q)$, $\alpha = 1, 2, 3$, by

$$\begin{aligned} &\cos\phi(q) = \sqrt{q}/2, && 0 < \phi < \pi/2, \\ &x_\alpha = \sin(\phi - v_\alpha)/\sin v_\alpha, && 0 < v_\alpha < \phi, \end{aligned} \tag{6}$$

and in terms of these variables

$$\mathrm{e}^{K_\alpha} = 1 + \tfrac{1}{2}[\sqrt{q(4-q)}\cot v_a - q], \qquad 0 < q < 4 \tag{7}$$

and the critical point (4) becomes

$$v_1 + v_2 + v_3 = 2\phi(q). \tag{8}$$

Baxter, Temperley and Ashley [6] evaluated the free energy at the critical point (4). For $0 < q < 4$, they obtained the expression

$$f_{\mathrm{critical}}^{\mathrm{RC}}(q) = \tfrac{1}{2}\ln q + \psi(\phi, v_1) + \psi(\phi, v_2) + \psi(\phi, v_3), \tag{9}$$

where

$$\psi(\phi, v) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sinh(\pi - \phi)x \sinh 2(\phi - v)x}{x \sinh \pi x \cosh \phi x}\,\mathrm{d}x, \qquad 0 < q < 4. \tag{10}$$

The integral (10) can be explicitly evaluated using contour integration by completing a contour in the upper-half complex z plane. The integrand has poles at

$$\begin{aligned} &\sinh(\pi z_1) = 0 \quad &&\text{or} \quad && z_1 = ni, && n = 1, 2, \ldots, \\ &\cosh(\phi z_2) = 0 \quad &&\text{or} \quad && z_2 = \frac{\pi}{2\phi}(2m+1)i, && m = 0, 1, 2, \ldots, \end{aligned}$$

and the evaluation of residues depends on whether z_1 and z_2 overlap. We have the following two cases:

Case 1. There is no overlapping between z_1 and z_2, namely, $\pi/2\phi$ is either irrational or

$$\frac{\pi}{2\phi} = \frac{M}{N}, \qquad M = 1, 2, 3, \ldots, \quad N < M \tag{11}$$

with N even. Then, both z_1, z_2 are simple poles and one obtains straightforwardly

$$\begin{aligned} \psi(\phi, v) = {} & \sum_{n=1}^{\infty} \frac{1}{n} \tan(n\phi) \sin 2n(\phi - v) \\ & + \sum_{m=0}^{\infty} \frac{2}{2m+1} \cot\left[\left(m + \frac{1}{2}\right)\frac{\pi^2}{\phi}\right] \sin\left[(2m+1)\frac{v\pi}{\phi}\right], \end{aligned} \tag{12}$$

where the two terms come from residues at z_1 and z_2, respectively.

Case 2. There is overlapping between z_1 and z_2. This occurs when $\pi/2\phi$ is given by (11) but with $N =$ odd. The residues now consist of three terms and one has

$$\psi(\phi, v) = R_1(\phi, v) + R_2(\phi, v) + R_3(\phi, v), \tag{13}$$

where R_1 is the residues from simple poles in z_1, R_2 is the residues from double poles and R_3 is the residues from simple poles in z_2, if any.

Again, the computation of residues can be carried out. After some algebra and manipulation, we find the results

$$\begin{aligned} R_1(\phi, v) = {} & \sum_{k=1}^{M-1} \tan(k\phi) \left[\sin(2k\phi) \int_{2v}^{\pi/2k} \frac{\cos(M-k)x}{\sin(Mx)}\, \mathrm{d}x \right. \\ & \left. - \cos(2k\phi) \int_0^{2v} \frac{\sin(M-k)x}{\sin(Mx)}\, \mathrm{d}x \right], \\ R_2(\phi, v) = {} & \frac{2(-1)^p}{MN\pi} \mathrm{Ti}_2(\tan u) + \frac{(-1)^p(N-p)}{MN} \ln\cot u, \\ R_3(\phi, v) = {} & -\frac{2M}{N} \sum_{k=1}^{(N-1)/2} \cot\left(\frac{2kM\pi}{N}\right) \int_0^{2v} \frac{\sin(2kMx/N)}{\sin(Mx)}\, \mathrm{d}x. \end{aligned} \tag{14}$$

Here, the number u and integer p in $R_2(\phi, v)$ are given by the parametrization

$$Mv = p\pi/2 + u, \qquad \text{with} \quad 0 < u < \pi/2, \quad p = 0, 1, 2, \ldots, \quad p < N, \tag{15}$$

and the function

$$\mathrm{Ti}_2(x) = \int_0^x \frac{\tan^{-1} t}{t}\, \mathrm{d}t = x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots \tag{16}$$

is the inverse tangent integral function [11].

It is instructive to see how the critical free energy passes from (13) to (12) as q varies and $\pi/2\phi$ changes from rational to irrational. Indeed, any irrational $\pi/2\phi$ can be reached by taking an appropriate $M, N \to \infty$ limit of $\pi/2\phi = N/M$. In this limit, we have $R_2 = 0$ by (14). It can be verified that R_1 and R_3 can be recast into equivalent forms

$$\begin{aligned}R_1 &= \sum_{n=1}^{M-1} \frac{1}{n} \tan(n\phi) \sin[2n(\phi - v)] + \sum_{n=1}^{M-1} \tan(n\phi) \bigg[\sin(2n\phi) \int_{2v}^{\pi/2n} \cot(Mx)\cos(nx)\, \mathrm{d}x \\ &\qquad + \cos(2n\phi) \int_0^{2v} \cot(Mx)\sin(nx)\, \mathrm{d}x \bigg], \\ R_3 &= 2 \sum_{m=0}^{(N-3)/2} \cot\left[\left(m+\frac{1}{2}\right)\frac{\pi x}{\phi}\right] \left\{ \frac{\sin[(2m+1)v\pi/\phi]}{2m+1} \right. \\ &\qquad \left. - \frac{M}{N}\int_0^{2v} \cot(Mx) \sin\left[\left(m+\frac{1}{2}\right)\frac{\pi x}{\phi}\right] \mathrm{d}x \right\}.\end{aligned} \tag{17}$$

In the large M, N limit, the integrals in (17) vanish. Then, R_1 and R_3 become the first and second terms, respectively, and one recovers (12).

For $q = 2$, we have $\phi = \pi/4$, $M = 2$, $N = 1$. Since N is odd, we combine (13) with (9) and after some algebra obtain the expression

$$f_{\text{critical}}^{\text{RC}}(2) = \frac{1}{2}\ln 2 + \sum_{\alpha=1}^{3}\left[\frac{1}{2}\ln(\cot v_\alpha) + \frac{1}{\pi}\mathrm{Ti}_2(\cot 2v_\alpha)\right], \qquad v_1 + v_2 + v_3 = \pi/2. \tag{18}$$

On the other hand, the $q = 2$ random cluster model is completely equivalent to an Ising model on the same triangular lattice with anisotropic interactions $K_\alpha/2$, $\alpha = 1, 2, 3$. Namely, we have

$$\begin{aligned}Z_{N_s}^{\text{Ising}} &= \sum_{\sigma=\pm1}\prod_{S} \mathrm{e}^{(K_\alpha/2)\sigma_i\sigma_j} \\ &= \mathrm{e}^{-N_s(K_1+K_2+K_3)/2} Z_N^{\text{RC}}(2).\end{aligned} \tag{19}$$

Also from (7) for $q = 2$, we have

$$\mathrm{e}^{K_\alpha} = \cot v_\alpha, \qquad \sinh K_\alpha = \cot(2v_\alpha). \tag{20}$$

It follows that the two free energies are related by

$$\begin{aligned}f^{\text{Ising}} &= f^{\text{RC}}(2) - \tfrac{1}{2}(K_1 + K_2 + K_3) \\ &= f^{\text{RC}}(2) - \tfrac{1}{2}\ln[(\cot v_1)(\cot v_2)(\cot v_3)],\end{aligned} \tag{21}$$

a relation which holds for *all* temperatures.

Now the Ising free energy is known [12] to be

$$\begin{aligned}f^{\text{Ising}} &= \ln 2 + \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \ln[\cosh K_1 \cosh K_2 \cosh K_3 + \sinh K_1 \sinh K_2 \sinh K_3 \\ &\qquad - \sinh K_1 \cos\theta - \sinh K_2 \cos\phi - \sinh K_3 \cos(\theta+\phi)]\, \mathrm{d}\theta\, \mathrm{d}\phi.\end{aligned} \tag{22}$$

It can be verified that the critical condition (8), or $v_1 + v_2 + v_3 = \pi/2$, is equivalent to

$$\cosh K_1 \cosh K_2 \cosh K_3 + \sinh K_1 \sinh K_2 \sinh K_3 = \sinh K_1 + \sinh K_2 + \sinh K_3, \tag{23}$$

which can also be written as

$$ab + bc + ca = 1, \tag{24}$$

with

$$a = \cot 2v_1 = \sinh K_1, \qquad b = \cot 2v_2 = \sinh K_2, \qquad c = \cot 2v_3 = \sinh K_3. \tag{25}$$

Thus, at the critical point, the Ising free energy assumes the form

$$f_{\rm critical}^{\rm Ising} = \ln 2 + \tfrac{1}{2} I(a,b,c), \tag{26}$$

where a, b, c are subject to (24) and $I(a,b,c)$ defined in (1). Also by combining (18) with (21), we have

$$f_{\rm critical}^{\rm Ising} = \frac{1}{2}\ln 2 + \frac{1}{\pi}[\mathrm{Ti}_2(a) + \mathrm{Ti}_2(b) + \mathrm{Ti}_2(c)]. \tag{27}$$

Equating the last two expressions, we obtain

$$I(a,b,c) = -\ln 2 + \frac{2}{\pi}[\mathrm{Ti}_2(a) + \mathrm{Ti}_2(b) + \mathrm{Ti}_2(c)], \qquad ab+bc+ca = 1. \tag{28}$$

For the integral $I(A,B,C)$ with A, B, C arbitrary, we introduce variables $a = AS$, $b = BS$, $c = CS$ with $S = 1/\sqrt{AB+BC+CA}$, so that (24) holds. Then, one has

$$\begin{aligned} I(A,B,C) &= -\ln S + I(a,b,c) \\ &= -\ln(2S) + \frac{2}{\pi}[\mathrm{Ti}_2(a) + \mathrm{Ti}_2(b) + \mathrm{Ti}_2(c)]. \end{aligned} \tag{29}$$

The last line in (29) establishes the integration formula give in the abstract.

It is readily checked that (29) yields the previously known values [13–16]

$$\begin{aligned} I(2,2,0) &= \frac{4}{\pi}\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} - \cdots\right), \\ I(2,2,2) &= \frac{6}{\pi}\mathrm{Ti}_2\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{2}\ln 3, \end{aligned} \tag{30}$$

which give, respectively, the numbers of spanning trees on large rectangular and triangular lattices. It is curious that $I(2,2,2)$ was first evaluated via the use of a $q = 0$ random cluster model [15].

For completeness, we give results for $q = 1$ ($\phi = \pi/3, M = 3, N = 2$) and $q = 3$ ($\phi = \pi/6, M = 3, N = 1$) obtained by using (9) and (12), and (9) and (13), respectively, as dictated by the value of N. After some algebra, we find

$$\begin{aligned} f^{\rm RC}(1) &= K_1 + K_2 + K_3, \\ f_{\rm critical}^{\rm RC}(3) &= \frac{1}{4}\ln\left(\frac{9}{8}\right) + \frac{3}{2}\ln\left(\frac{2+\sqrt{3}}{2}\right) + \sum_{\alpha=1}^{3}\left[\frac{1}{6}\ln\left(\frac{\sqrt{3}\cot v_\alpha - 1}{\sqrt{3}\tan v_\alpha - 1}\right)\right. \\ &\quad \left. + \frac{1}{2}\ln(1+\sqrt{3}\cot 2v_\alpha) + \frac{2}{3\pi}\mathrm{Ti}_2(\cot 3v_\alpha)\right], \qquad v_1 + v_2 + v_3 = \pi/3. \end{aligned} \tag{31}$$

The $q = 1$ expression is the same as that computed directly from (2) which holds for all temperatures.

In summary, we have obtained the free energy (9) of the random cluster model at its critical point for $0 < q < 4$ in the form of a series. The resulting expression is given by (12) if $\pi/2\phi(q) = \pi/2\cos^{-1}(\sqrt{q}/2)$ is irrational or a rational number with an even denominator, and is given by (13) if $\pi/2\phi$ is a rational number with an odd integer in the denominator. For $q = 2$, our result leads to a closed-form evaluation of the integral $I(A,B,C)$ given by (1) for general A, B, C. We mention in passing that Bazhanov and Stroganov [17] have considered the free energy of the free-fermion model [18], which is of the form of a double integral more general than that of $I(A,B,C)$. They obtained a closed-form expression in terms of derivatives of Jacobi elliptic functions.

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